



TITLE:

Weakly k -linked graphs(Topological Graph Theory)

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Weakly k -linked graphs

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1. はじめに

多重辺は含んでもよいが、ループは含まない有限無向グラフを考える。 G をグラフとし、 $V(G)$ は G の点の集合、 $E(G)$ は G の辺の集合とする。パスまたはサイクルは、1つの辺を高々1回しか通れないが、同じ点を2回以上通ってもよいことにする。グラフ G の辺連結度を $\lambda(G)$ であらわす。

定義 1. G が weakly k -linked

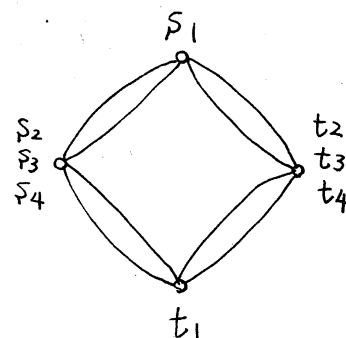
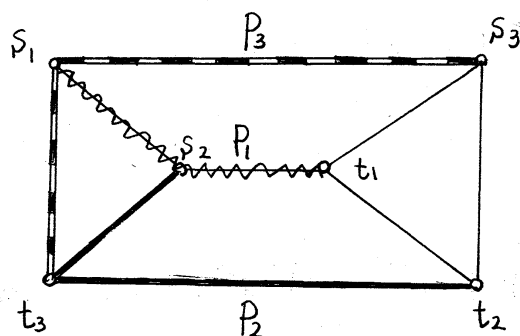
$\iff G$ の任意の長組の点の対 (重複点を含んでもよい) $(s_1, t_1), \dots, (s_k, t_k)$ に対して、辺素な (i.e. 互いに辺を共有しない) パス P_1, P_2, \dots, P_k があって、 P_i は s_i と t_i を結ぶ ($1 \leq i \leq k$)。

定義 2. $g(k) := \min \{ n \mid \text{もし } \lambda(G) \geq n \Rightarrow G \text{ は weakly } k\text{-linked} \}$

次の不等式がすぐに導ける。

$$(1.1) \quad g(k) \geq \begin{cases} k & (k \text{ が odd}) \\ k+1 & (k \text{ が even}) \end{cases}.$$

例 1.

 $\lambda(G)=3$, G は weakly 3-linked. $\lambda(G)=4$, G は weakly 4-linked ではない

次の二ことが既知である。

(1.2) $g(2) = g(3) = 3$ (Okamura [4]), $g(4) = 5$ (Hirata, Kubota, Paito [1], Mader [3]), $g(2k+1) \leq 3k$, $g(2k) \leq 3k-1$ ($k \geq 2$) (Okamura [8]).

次の結果がえられた。

定理 1. $g(3k) \leq 4k$, $g(3k+1) \leq g(3k+2) \leq 4k+2$ ($k \geq 2$)

定理 1 を証明するために、定理 2 が必要である。

定理 2. $k \geq 4$ は偶数, G は k -辺連グラフ, $\{s, t\} \in V(G)$, $f \in E(G)$ とする。このとき、 s と t を結ぶ f を通らないパス P があって、 $\lambda(G - E(P) - f) \geq k-2$ 。

関連する話題が、[5], [6], [7] でも議論されている。

Notations and definitions

Let $X, Y, (x, y) \in V(G)$, $f \in E(G)$ and $X \cap Y = \emptyset$. We often denote (x) by x . $V(f)$ denotes the set of end vertices of f . We denote by $\partial(X, Y; G)$ the set of edges with one end in X and the other in Y , and set $\partial(X; G) := \partial(X, V(G) - X; G)$, $e(X, Y; G) := |\partial(X, Y; G)|$ and $e(X; G) := |\partial(X, V(G) - X; G)|$. $\lambda(x, y; G)$ denotes the maximal number of edge-disjoint paths between x and y . We set $\bar{X} := V(G) - X$, $N(x; G) := \{a \in V(G) - x \mid e(a, x) > 0\}$, $N(X; G) := \bigcup_{x \in X} N(x; G)$, and $\Gamma(G, k) := \{Z \subset V(G) \mid \text{for each } a, b \in Z, \lambda(a, b; G) \geq k\}$. In all notations, we often omit G . G/X denotes the graph obtained from G by contracting X , and for $a \in X$, we denote the corresponding vertex in G/X by \tilde{a} . A path $P = P[x, y]$ denotes a path between x and y , and for $a, b \in V(P)$, $P(a, b)$ denotes a subpath of P between a and b . We call $X \subset V(G)$ a k -set if $|X| \geq 2$, $|\bar{X}| \geq 2$ and $e(X) = k$, and a k -set X is called minimum if for each $Y \subset X$ with $|Y| \geq 2$, $e(Y) \geq k+1$. For $a, b \in N(x)$ with $a \neq b$, $f \in \partial(x, a)$ and $g \in \partial(x, b)$, $G_x^{a, b}$ denotes the graph $(V(G), (E(G) \cup h) - \{f, g\})$, where h is a new edge between a and b and is called a lifting of G at x arising from the lifting of f and g at x . We call $G_x^{a, b}$ admissible if for each $y, z \in V(G) - x$ with $y \neq z$, $\lambda(y, z; G_x^{a, b}) = \lambda(y, z; G)$.

2. Preliminaries

In this section we assume that $k \geq 1$ is an integer and G is a graph.

Lemma 2.1 (Mader [3] and [5]). If $k \geq 2$, $\lambda(G) \geq k$, $s \in V(G)$, and $(f_1, f_2) \subset \partial(s)$, then there exists a cycle C such that $(f_1, f_2) \subset E(C)$ and $\lambda(G - E(C)) \geq k-2$.

Lemma 2.2 (Mader [2]). If $x \in V(G)$, $e(x) \geq 4$, $|N(x)| \geq 2$, and x is not a cut-vertex, then there exists an admissible lifting of G at x .

Lemma 2.3 ([8, Lemma 3]). If $k \geq 3$, $V(G) = W_1 \cup W_2$, $W_1 \cap W_2 = \emptyset$, $W_1 \in \Gamma(G, k)$, and each $x \in W_2$ has even degree, then we can obtain a k -edge-connected graph $G(W_1, k)$ from G such that $W_1 \subset V(G(W_1, k))$ by sequences of vertex-deletions and edge-liftings.

Lemma 2.4 ([8]). If $k \geq 4$ is even, $\lambda(G) \geq k$, and $s, t, a \in V(G)$ ($s=t$ or $s \neq t$), then there exists a path $P[s, t]$ such that $a \in V(P)$ and $\lambda(G - E(P)) \geq k-2$.

Lemma 2.5. If $X \subset V(G)$, $e(X) = k$, and $\lambda(G/X) = \lambda(G/\bar{X}) = k$, then $\lambda(G) = k$.

Lemma 2.6. If $\lambda(G) \geq k$, $X, Y \subset V(G)$, $X-Y$, $Y-X$, $X \cap Y$, and $\overline{X \cup Y}$ are not empty, and $e(X) = e(Y) = k$, then k is even and $e(X-Y) = e(Y-X) = e(X \cap Y) = k$.

Proof. By simple counting we have

$$e(X-Y) + e(Y-X) = e(X) + e(Y) - 2e(X \cap Y, \overline{X \cup Y}),$$

$$e(X \cap Y) + e(X \cup Y) = e(X) + e(Y) - 2e(X-Y, Y-X).$$

Thus $e(X-Y) = e(Y-X) = e(X \cap Y) = k$, and $k = e(X) = e(X-Y) + e(X \cap Y) \equiv 0 \pmod{2}$.

Lemma 2.7. Suppose that $\lambda(G) = k \geq 3$ and $|V(G)| \geq 4$. Then

- (1) If k is odd, G is k -regular, and $x \in V(G)$, then $|N(x)| \geq 3$.
- (2) if k is even, $(x, y) \subset V(G)$, $e(x) = k$, and $e(y) \leq k+1$, then $e(x, y) \leq k/2$.

Proof. (1) If $|N(x)| \leq 2$, then for some $y \in N(x)$,

$e(x,y) \geq (k+1)/2$ and $e((x,y)) \leq k-1$.

(2) If $e(x,y) \geq k/2+1$, then $e((x,y)) = e(x) + e(y) - 2(k/2+1) \leq k-1$.

3. Proof of Theorem 2

If $V(f) = (s, t)$, then by Lemma 2.1, for a $g \in \partial(s) - f$, G has a cycle C such that $(f, g) \in E(C)$ and $\lambda(G - E(C)) \geq k-2$, and the result holds. If $V(f) = (a, s)$ and $a \neq t$, then by Theorem 1(1) in

[5] G has a path $P[a, t]$ such that $f \in E(P)$ and $\lambda(G - E(P)) \geq k-2$.

Thus let $V(f) \cap (s, t) = \emptyset$, and set $T := V(f) \cup (s, t)$. We may assume (see the proof of Theorem 2 and Figures 2, 3 in [8])

(3.2) For each $x \in T$, $e(x) = k$, and for each $x \in V(G)$, $e(x) = k$ or $k+1$.

We proceed by induction on $|E(G)|$. We assume that the result does not hold in G . Then

(3.3) $e(s, t) = 0$.

(3.4) $V(G) - T \neq \emptyset$.

Proof. Assume $V(G) = T$. Let $V(f) = (a, b)$. Then

$$2e(s, t) = e(s) + e(t) - e((s, t)) = e(a) + e(b) - e((a, b)) = 2e(a, b) > 0.$$

(3.5) If $X \subset V(G) - T$ and $|X| \geq 2$, then $e(X) \geq k+1$.

Proof. Assume $e(X) = k$ and $x \in X$. By induction G/X has a required path $P[s, t]$. If $\tilde{x} \notin V(P)$, then P is a required path for G , thus let $\tilde{x} \in V(P)$ and $E(P) \cap \partial(\tilde{x}; G/X) = (g_1, g_2)$. By Lemma 2.1 G/\bar{X} has a cycle C such that $(g_1, g_2) \in E(C)$ and $\lambda(G/\bar{X} - E(C)) = k-2$. By combining P and C in G , we get a required path for G (see Lemma 2.5).

(3.6) If $x \in V(G) - T$, then $e(x) = k+1$.

Proof. Assume $e(x) = k$. By Lemma 2.2 there is an admissible lifting G_x of G at x . Set $G_1 := G_x(V(G_x) - x, k)$ (see Lemma 2.3). Then $\lambda(G_1) = k$, and by induction G_1 has a required path $P[s, t]$. Let P_1 be the corresponding path in G , and let P_2 be a simple subpath of P_1 between s and t . Then P_2 is a required path for G .

(3.7) If $(x, y) \in V(G) - T$ and $g \in \partial(x, y)$, then $\lambda(G - g) < k$.

(3.8) If $X \subset V(G)$ is a minimum k -set and $(x, y) \in X - T$, then $e(x, y) = 0$.

Proof. Assume $e(x, y) > 0$. By (3.7) there is a k -set Y such that $|Y \cap (x, y)| = 1$. Then $Y - X \neq \emptyset \neq \overline{X} - Y$, since X is minimum. Then by Lemma 2.6 $e(X \cap Y) = k$. Thus $X \cap Y = \{x\}$ or $\{y\}$, contrary to (3.6).

(3.9) If $X \subset V(G)$ is a k -set, then $|X \cap T| = 2$.

Proof. Let $X_1 \subset X$ be a minimum k -set (X_1 might equal X). By (3.5) $|X_1 \cap T| \geq 1$. Assume $X_1 \cap T = \{a\}$. By (3.1), (3.2), and (3.6) $|X_1 - a| \geq 2$. Let $x \in X_1 - a$, $y \in \overline{X}_1$, and set $G_1 := G/\overline{X}_1$. Then $|V(G_1)| \geq 4$ and by (3.8) $N(x; G_1) \subset \{a, \tilde{y}\}$. By Lemma 2.7(2) $e(x, a) \leq k/2$ and $e(x, \tilde{y}; G_1) \leq k/2$, contrary to (3.6). Thus $|X_1 \cap T| \geq 2$, and similarly $|\overline{X} \cap T| \geq 2$. Hence $|X \cap T| = 2$.

(3.10) G has no k -set.

Proof. Assume that G has a k -set X . Let $X_1 \subset X$ and $X_2 \subset \overline{X}$ be

minimum k -sets. By (3.9) $|X_i \cap T| = 2$ ($i=1,2$).

(3.10.1) If Y is a k -set and $X_1 \cap Y \neq \emptyset$, then $X_1 \subset Y \subset \bar{X}_2$.

For, if $X_1 - Y \neq \emptyset$, then $Y - X_1 \neq \emptyset$ and $\bar{Y} \cap \bar{X}_1 \neq \emptyset$, since X_1 is minimum. By Lemma 2.6 $e(X_1 - Y) = e(X_1 \cap Y) = e(Y - X_1) = k$.

Thus $|X_1| = 2$ and by (3.6) $X_1 \subset T$. Let $X_1 = (a_1, a_2)$ and $a_1 \in Y$. If $|Y - X_1| \geq 2$, then by (3.9) $|(Y - X_1) \cap T| = 2$, and so $|Y \cap T| = 3$, contrary to (3.9). Thus $|Y - X_1| = 1$, and by (3.6) $Y - X_1 \subset T$. Let $Y - X_1 = (a_3)$. $Y \cap X_2 = (a_3)$, and so $X_2 - a_3 \in T$ as above, let $X_2 - a_3 = (a_4)$. Now $e(a_1, a_2) = e(a_1, a_3) = e(a_3, a_4) = k/2$. $e(\bar{T}) \leq k$ and by (3.4) $\bar{T} \neq \emptyset$, contrary to (3.6) or (3.9). Hence $X_1 \subset Y$. If $Y \cap X_2 \neq \emptyset$, then similarly $X_2 \subset Y$, contrary to (3.9). Therefore $Y \subset \bar{X}_2$.

(3.10.2) $V(G) = X_1 \cup X_2$.

For, assume $V(G) \neq X_1 \cup X_2$. Then there is a $Y \subset \bar{X}_2$ such that $X_1 \not\subset Y$ and $e(Y) = k$. We choose Y such that $|Y|$ is minimal (Y might equal \bar{X}_2). Let $x \in Y - X_1$. If $N(x) \subset X_1 \cup \bar{Y}$, then $e(x, X_1) \geq (k+1)/2$ or $e(x, \bar{Y}) \geq (k+1)/2$, and so $e(X_1 \cup x) < k$ or $e(Y - x) < k$. Thus for some $y \in Y - X_1$, $e(x, y) > 0$. By (3.7) there is a k -set Z such that $|Z \cap (x, y)| = 1$. We may let $X_1 \cap Z \neq \emptyset$ (if not, then we take \bar{Z} as Z). Then by (3.10.1) $X_1 \subset Z \subset \bar{X}_2$. By choice of Y , $Z - Y \neq \emptyset$. By Lemma 2.6 $e(Z - Y) = k$, contrary to (3.6) or (3.9).

Let $X_1 \cap T = (a_1, a_2)$ and $X_2 \cap T = (b_1, b_2)$. By (3.8) for each $x \in X_1 - T$, $N(x) \subset (a_1, a_2) \cup X_2$, and for each $y \in X_2 - T$, $N(y) \subset (b_1, b_2) \cup X_1$. By (3.6), for $i=1,2$, $|X_i|$ is even and $(k+1)|X_i - T| \leq 3k$, thus $|X_i - T| = 0$ or 2 . By (3.4) we may let $X_1 - T = (x_1, x_2)$. For $i=1,2$, if $e(a_i, X_2) \geq k/2$, then $e(X_2 \cup a_i) \leq k$, thus $e(a_i, X_2) \leq k/2 - 1$. Similarly

$e(x_i, X_2) \leq k/2$ ($i=1,2$). If $V(f) = \{b_1, b_2\}$, then $e(a_1, a_2) = 0$. By Lemma 2.7(2) $e(a_1, x_i) \leq k/2$ ($i=1,2$), and so $e(a_1, x_i) > 0$ ($i=1,2$). Similarly $e(a_2, x_i) > 0$ ($i=1,2$) and the result follows. If $V(f) = \{a_1, a_2\}$, then we may let $|X_2 - T| = 0$, contrary to (3.3). Thus we may let $V(f) = \{a_1, b_1\}$. Now $e(a_2, b_2) = 0$. If $e(b_2, (x_1, x_2)) > 0$, let $g_1 \in \partial(b_2, x_1)$, then $e(x_1, a_2) = 0$, and so there are $g_2 \in \partial(x_1, x_2)$ and $g_3 \in \partial(a_2, x_2)$. $e((a_1, x_1, a_2); G/X_2) = e(a_1) + e(x_1) + e(a_2) - 2e(a_1, (x_1, a_2)) \geq 3k + 1 - 2(k-1) = k+3$. Thus $\lambda(G/X_2 - (f, g_1, g_2, g_3)) \geq k-2$. Therefore $e(b_2, (x_1, x_2)) = 0$, and so $|X_2 - T| = 2$ (note that $e(b_2, a_2) = 0$). Let $X_2 - T = \{y_1, y_2\}$. Similarly $e(a_2, (y_1, y_2)) = 0$. Since $\{a_1, b_1\}$ is not a separating set, $e((x_1, x_2), (y_1, y_2)) > 0$. Let $g_1 \in \partial(x_1, y_1)$. Then $e(x_1, a_2) = e(y_1, b_2) = 0$, and for $g_2 \in \partial(x_1, x_2)$, $g_3 \in \partial(x_2, a_2)$, $g_4 \in \partial(y_1, y_2)$ and $g_5 \in \partial(y_2, b_2)$, $\lambda(G - (f, g_1, g_2, g_3, g_4, g_5)) = k-2$.

By (3.7) and (3.10) for each $x \in V(G) - T$, $N(x) \subset T$. Let $V(f) = \{a_1, a_2\}$. $k|\bar{T}| \leq 4k-2$ and $|\bar{T}|$ is even, thus and by (3.4) $|\bar{T}| = 2$. Let $\bar{T} = \{x_1, x_2\}$. By (3.10) $e(s, a_i) < k/2$ ($i=1,2$), and so by (3.3) $e(s, (x_1, x_2)) > 0$ and $e(t, (x_1, x_2)) > 0$. We may let $e(s, x_1) > 0$, then $e(t, x_1) = 0$, $e(t, x_2) > 0$ and $e(s, x_2) = 0$. $\{a_1, a_2\}$ is not a separating set, thus $e((s, x_1), (t, x_2)) > 0$, and so there is a $g_1 \in \partial(x_1, x_2)$. For $i=1,2$, $e((s, x_2, a_i)) \geq 3k - 2e(a_i, (s, x_2)) \geq 3k - 2(k-1) = k+2$. Thus for $g_2 \in \partial(s, x_1)$ and $g_3 \in \partial(t, x_2)$, $\lambda(G - (f, g_1, g_2, g_3)) \geq k-2$.

3. Proof of Theorem 1

The proof of Lemma 4.1 will be given later.

Lemma 4.1. Suppose that $k \geq 4$ is an even integer, $n \geq 3$ is an integer, G is a 2-connected graph, $V(G) = T \cup W_1 \cup W_2$ (disjoint union), $T = \{s_1, \dots, s_n, t_1, \dots, t_n\}$, $|T| = 2n$, $T \cup W_1 \in \Gamma(G, k)$, $e(s_i) = e(t_i) = k$ ($1 \leq i \leq n$), for each $x \in W_1$, $e(x) = k$ or $k+1$, and for each $x \in W_2$, $e(x) < k$ is even. Then there is a subgraph $G^* \subset G$ such that

- (a) for some $1 \leq i < j < 1 \leq n$, $G - E(G^*)$ has edge-disjoint paths $P_1[s_i, t_i]$, $P_2[s_j, t_j]$ and $P_3[s_1, t_1]$.
- (b) $V(G^*) = K_1 \cup K_2$ and $K_1 \cap K_2 = \emptyset$,
- (c) $T - \{s_i, t_i, s_j, t_j, s_1, t_1\} \subset K_1 \in \Gamma(G^*, k-4)$,
- (d) for each $x \in K_2$, $e(x; G^*)$ is even.

Proof of Theorem 1

By (1.1) it suffices to prove $g(3k) \leq 4k$ and $g(3k+2) \leq 4k+2$ ($k \geq 2$). Let $\alpha = 0$ or 1 , $m \geq 2$ is an integer, $k := 4m + 2\alpha$ and $n := 3m + 2\alpha$. Assume that G is a k -edge-connected graph and $\{s_1, \dots, s_n, t_1, \dots, t_n\} := T$ are vertices of G (not necessarily distinct). We prove that there are edge-disjoint paths P_1, \dots, P_n such that P_i joins s_i and t_i ($1 \leq i \leq n$). We may assume (see the proof of Theorem 2 and Figure 3 in [6])

(4.1) $e(s_i) = e(t_i) = k$ ($1 \leq i \leq n$) and for each $x \in V(G)$, $e(x) = k$ or $k+1$.

We proceed by induction on $|E(G)|$. If $s_1 = s_2$, then by Lemma 2.4 there is a path $P[t_1, t_2]$ such that $s_1 \in V(P)$ and $\lambda(G - E(P)) \geq k-2$. By induction $G - E(P)$ has edge-disjoint paths $P_3[s_3, t_3], \dots, P_n[s_n, t_n]$. Thus let $|T| = 2n$. By Lemma 4.1 there is a subgraph $G^* \subset G$ such that (a), (b), (c) and (d) hold. By Lemma 2.3

$G^*(K_1, k-4)$ is $(k-4)$ -edge-connected, and by induction $G^*(K_1, k-4)$ has $(n-3)$ edge-disjoint paths joining (s_r, t_r) ($1 \leq r \leq n$, $r \neq i, j, 1$). Thus the result holds in G .

Proof of Lemma 4.1

Suppose that G satisfies the hypothesis of Lemma 4.1, but the result does not hold. Choose G with this property such that $|E(G)|$ is minimal.

$$(4.2) \quad W_2 = \emptyset.$$

Proof. Assume $x \in W_2$. Then $e(x) \geq 4$. By Lemma 2.2 we have an admissible lifting G_x of G at x . The result holds in G_x , and so in G .

(4.3) If $1 \leq i < j \leq n$, $G_1 \subset G$ is a subgraph such that $G - E(G_1)$ has edge-disjoint paths $P_1[s_i, t_i]$ and $P_2[s_j, t_j]$, $V(G_1) = K_1 \cup K_2$, $K_1 \cap K_2 = \emptyset$, $T - (s_i, t_i, s_j, t_j) \subset K_1$, and for each $x \in K_2$, $e(x; G_1)$ is even, then $K_1 \notin \Gamma(G_1, k-2)$.

Proof. Assume $K_1 \in \Gamma(G_1, k-2)$. Let $1 \leq l \leq n$ and $l \neq i, j$. By Lemma 2.4 G_1 has a path $P[s_l, t_l]$ such that $\lambda(G_1 - E(P)) \geq k-4$. Let $G^* := G_1 - E(P)$.

$$(4.4) \quad \text{If } x \in W_1, \text{ then } e(x) = k+1.$$

Proof. Assume $e(x) = k$. By Lemma 2.2 there is an admissible lifting G_x of G at x . The result holds in G_x with $V(G_x) = T \cup (W_1 - x) \cup \{x\}$, and it also holds for G .

$$(4.5) \quad \text{If } x, y \in W_1 \text{ and } f \in \theta(x, y), \text{ then } \lambda(G - f) \leq k-1.$$

(4.6) If $a, b \in T$, then $e(a, b) = 0$.

Proof. If $f \in \partial(s_1, t_1)$, then by Theorem 2 there is a path $P[s_2, t_2]$ such that $f \notin E(P)$ and $\lambda(G - E(P) - f) \geq k - 2$, contrary to (4.3). If $f \in \partial(s_1, s_2)$, then by Theorem 2 G has a path $P_1[s_3, t_3]$ such that $f \notin E(P_1)$ and $\lambda(G - E(P_1) - f) \geq k - 2$. By Lemma 2.4 $G - E(P_1) - f$ has a path $P_2[t_1, t_2]$ such that $s_1 \in V(P_2)$ and $\lambda(G - E(P_1 \cup P_2) - f) \geq k - 4$.

(4.7) If $x_1, x_2 \in W_1$, $a_1, a_2 \in T$, $f_i \in \partial(x_i, a_i)$ ($i = 1, 2$) and $g \in \partial(x_2, a_2)$, then $V(G) - a_1 \in \Gamma(G - (f_1, f_2), k)$.

Proof. Set $G_1 := G - (f_1, f_2)$, and assume $V(G) - a_1 \in \Gamma(G_1, k)$. Set $G_2 := G_1 - (V(G) - a_1, k)$. If $a_1 = s_1$ and $a_2 = t_1$, then by Theorem 2 G_2 has a path $P[s_2, t_2]$ such that $g \notin E(P)$ and $\lambda(G_2 - E(P) - g) \geq k - 2$, contrary to (4.3). If $a_1 = s_1$ and $a_2 = s_2$, then by Theorem 2 G_2 has a path $P_1[s_3, t_3]$ such that $g \notin E(P_1)$ and $\lambda(G_2 - E(P_1) - g) \geq k - 2$. By Lemma 2.4 $G_2 - E(P_1) - g$ has a path $P_2[t_1, t_2]$ such that $a_2 \in V(P_2)$ and $\lambda(G_2 - E(P_1 \cup P_2) - g) \geq k - 4$.

(4.8) G has no k -set.

Proof. Assume X is a minimum k -set. Let $u \in \bar{X}$. If $x, y \in X \cap W_1$ and $f \in \partial(x, y)$, then $V(G) - (x, y) \in \Gamma(G - f, k)$. For, if not, then for some k -set Z , $|Z \cap (x, y)| = 1$. Then $Z - X \neq \emptyset \neq \overline{X \cup Z}$ and by Lemma 2.6 $e(X - Z) = e(X \cap Z) = k$. Thus $|X| = 2$ and $e(x) = e(y) = k$, contrary to (4.6). Thus by (4.5) $N(X \cap W_1; G/\bar{X}) \subset T \cup \{\bar{u}\}$, and by (4.6) $X \cap W_1 \neq \emptyset \neq X \cap T$. By (4.4) $|X \cap W_1| \geq 2$, and so $|X \cap T| \geq 2$. Let $a \in X \cap T$. Since $e(a, \bar{X}) < k/2$ (otherwise $e(X - a) \leq k$), by Lemma 2.2 for some $x, y \in N(a) \cap X$, $G_a^{x, y}$ is admissible. By (4.6) $(x, y) \in W_1$.

Let $f_1 \in \partial(a, x)$ and $f_2 \in \partial(a, y)$, then $V(G) - a \in \Gamma(G - (f_1, f_2), k)$. Let $b \in ((N(x) \cup N(y)) \cap X) - a$, then $b \in T$, contrary to (4.7).

By (4.2), (4.5), (4.6) and (4.8) G is a bipartite graph with the partition (T, W_1) . Let $a \in T$. By Lemma 2.2 for some $x, y \in N(a)$, $G_a^{x,y}$ is admissible and we can deduce a contradiction (see the proof of (4.8)).

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